Proceedings of the American Academy of Arts and Sciences.

VOL. 63. NO. 11.—MARCH, 1929.

TABLES OF LAGRANGEAN COEFFICIENTS FOR INTERPOLATING WITHOUT DIFFERENCES.

By EDWARD V. HUNTINGTON.

TABLES OF LAGRANGEAN COEFFICIENTS FOR INTERPOLATING WITHOUT DIFFERENCES.*

By Edward V. Huntington.

Introduction.

The tables presented in this paper are intended to facilitate the process of high-order interpolation in any given table in which the argument advances by equal intervals, provided a computing machine is available.

The tables in Part I are for use in interpolating for single or isolated values. Those in Part II are for use in continuous or systematic interpolation. In either case the result is obtained directly from the given tabulated values, without the necessity of forming columns of differences. Unless the columns of differences are desired for other purposes, the saving of time thus effected will be found to be substantial.

The tables are based on the well-known formula of Lagrange (see Appendix below) and are believed to be the first of their kind which are sufficiently extensive for practical use.** The figures given are exact in all cases, except in Tables 5, 5a, and 5b; in these tables, the entries in every tenth line are exact, the other entries being correct to eight places of decimals. For most work, a smaller number of decimals will of course be sufficient.

* Presented to the American Academy of Arts and Sciences, Boston, November 14, 1928. (An abstract of this paper was presented to the American Mathematical Society, April 28, 1923.)

** A very brief table of this sort is given by J. Burn and E. H. Brown in *Elements of Finite Differences*, 1915, page 31. Excellent tables for use with the formulas of Newton, Stirling, Bessel, Everett, etc., are available in many places (see, for example, J. W. Glover's *Tables of Applied Mathematics*, 1923); but all these formulas require the preliminary computation of the columns of differences. An extended study of the whole theory will be found in Karl Pearson's Tracts for Computers, No. II, On the Construction of Tables and on Interpolation, Part I, Uni-variate Tables (Cambridge University Press, 1920). The coefficients in Pearson's formulas v to xii ("Four-point Mid-panel Lagrangians"), on page 31, are the same as the entries opposite $n = 0.6$, 0.8, 0.4, 0.2 in the present Tables 3 and 3a; and the entries in Pearson's Table A ("Six-ordinate Mid-panel Lagrangians"), facing page 45, are the same as the entries opposite $n = 0.2$, 0.4, 0.6, 0.8 in the present Tables 5, 5a, and 5b. Pearson also gives, at similar intervals, a Table B for "Eight-Entry Lagrangians" and a Table C for "Ten Entry Lagrangians," facing page 46.

Part I. Single Interpolations.

Let . . . Y_{-1}, Y_0, Y_1, Y_2, . . . be a given set of tabulated values, corresponding to the given arguments . . . X_{-1}, X_0, X_1, X_2, . . . (these latter values being supposed to advance by equal intervals); and let it be required to find by interpolation the value Y_n corresponding to any given value X_n between X_0 and X_1. (Here n is any positive fraction between 0 and 1.) If 3rd degree interpolation is sufficiently accurate, use table 3 or 3a; if 5th degree interpolation is necessary, use table 5 or 5a or 5b; the process being a very simple one, as follows:

*Table 3. Third degree interpolation in main part of table.**

If the interval between X_0 and X_1 lies in the main part of the given table (so that values on both sides of Y_0 and Y_1 are known), find from table 3 the factors A_{-1}, A_0, A_1, A_2 corresponding to the given value of n. Multiply these factors by the given values Y_{-1}, Y_0, Y_1, Y_2 respectively, and add the products. The result will be Y_n, according to the formula

$$Y_n = A_{-1}Y_{-1} + A_0Y_0 + A_1Y_1 + A_2Y_2$$

(where the A's are functions of n). The computing machine will accumulate the products as formed—the positive ones by addition, the negative ones by subtraction—so that nothing need be written down except the final result, Y_n.

To test the precision of the result of this 3rd degree interpolation, first find the value of $D_4 = Y_{-1} - 4Y_0 + 6Y_1 - 4Y_2 + Y_3$. Then the correction E_3 which would have to be added to the result of the 3rd degree interpolation to make it equal the result of a 4th degree interpolation will certainly lie between 0 and 0.024 D_4.

[More precisely, $E_3 = (n + 1)_4 D_4$, where

$$(n + 1)_4 = (n + 1)\ (n)\ (n - 1)\ (n - 2)/(4!).$$

From a table of binomial coefficients it is seen that the values of $(n + 1)_4$ for values of n between 0 and 1 will be positive and less than 0.024.]

* Here by "third degree interpolation" is meant interpolation by means of a polynomial of the third degree, the coefficients being so chosen that the curve representing the polynomial passes exactly through the four points nearest the interval in question; in this case through the four points (X_{-1}, Y_{-1}), (X_0, Y_0), (X_1, Y_1), (X_2, Y_2). Similarly, a "fifth degree interpolation" means interpolation by a polynomial of the fifth degree, passing exactly through six points.

Table 3a. Third degree interpolation at beginning of table.

If the interval from X_0 to X_1 is the first interval in the given table, the value Y_{-1} is not known, and Table 3 cannot be used. In this case, find the factors a_0, a_1, a_2, a_3, from Table 3a, and compute the required value Y_n from the formula

$$Y_n = a_0 Y_0 + a_1 Y_1 + a_2 Y_2 + a_3 Y_3.$$

To test the precision in this case, first find $d_4 = Y_0 - 4Y_1 + 6Y_2 - 4Y_3 + Y_4$. Then the correction e_3 which would have to be added to the result of the 3rd degree interpolation to make it equal to the result of a 4th degree interpolation will lie between 0 and $-0.042\ d_4$.

[More precisely, $e_3 = (n)_4 d_4$ where

$$(n)_4 = (n)\ (n-1)\ (n-2)\ (n-3)/4!]$$

Note.—If the interval between X_0 to X_1 is the *last* interval in the given table, we have only to imagine the table written in the reverse order (so that X_1 becomes X_0, X_0 becomes X_1, etc.) and to use $1 - n$ in place of n.

Table 5. Fifth degree interpolation in main part of table.

If fifth degree interpolation is required, and the interval between X_0 and X_1 lies in the main part of the given table, we have

$$Y_n = B_{-2}Y_{-2} + B_{-1}Y_{-1} + B_0 Y_0 + B_1 Y_1 + B_2 Y_2 + B_3 Y_3,$$

where the factors B_{-2}, B_{-1}, B_0, B_1, B_2, B_3, are taken from Table 5.

The correction E_5 which would have to be added to the result of this fifth degree interpolation to make it equal to the result of a sixth degree interpolation lies between 0 and $-0.005\ D_6$, where $D_6 = Y_{-2} - 6Y_{-1} + 15Y_0 - 20Y_1 + 15Y_2 - 6Y_3 + Y_4$.

[More precisely, $E_5 = (n+2)_6 D_6$, where

$$(n+2)_6 = (n+2)\ (n+1)\ (n)\ (n-1)\ (n-2)\ (n-3)\ /\ 6!]$$

Table 5a. Fifth degree interpolation at beginning of table.

If the interval between X_0 and X_1 is the first interval in the table, we have

$$Y_n = b_0 Y_0 + b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y_4 + b_5 Y_5,$$

where the factors b_0, b_1, b_2, b_3, b_4, b_5, are taken from Table 5a.

The "correction" e_5 now lies between 0 and $-0.024\ d_6$ where

$$d_6 = Y_0 - 6Y_1 + 15Y_2 - 20Y_3 + 15Y_4 - 6Y_5 + Y_6$$

[More precisely, $e_5 = (n)_6 d_6$, where

$$(n)_6 = (n)\ (n-1)\ (n-2)\ (n-3)\ (n-4)\ (n-5)/6!]$$

Table 5b. Fifth degree interpolation in second interval of table.

If the interval between X_0 and X_1 is the second interval in the table, we have

$$Y_n = c_1 Y_{-1} + c_0 Y_0 + c_1 Y_1 + c_2 Y_2 + c_3 Y_3 + c_4 Y_4$$

where the factors c_{-1}, c_0, c_1, c_2, c_3, c_4 are taken from Table 5b.

The "correction" e_5' now lies between 0 and 0.007 d_6', where

$$d_6' = Y_{-1} - 6Y_0 + 15Y_1 - 20Y_2 + 15Y_3 - 6Y_4 + Y_5$$

[More precisely, $e_5' = (n+1)_6 d_6'$, where

$$(n+1)_6 = (n+1)\ (n)\ (n-1)\ (n-2)\ (n-3)\ (n-4)/6!]$$

Note. If the interval in question is the last, or next to the last, interval in the table, we have only to imagine the table written in the reverse order, and to use $1 - n$ in place of n.

Part II. Continuous Interpolation.

Let it be required to expand a given table, by reducing the tabular interval to 1/10 of its size.

It will be found desirable to begin by interpolating a mid-value in each interval, using Table 3 or 3a (for third degree interpolation) or Table 5 or 5a or 5b (for fifth degree interpolation), with $n = 0.50$, and then to subdivide the resulting intervals into fifths.*

In subdividing into fifths, it is well to use a cumulative process (Table 30 or 30a, or Table 50 or 50a or 50b), in which each result is left in the machine to serve as the starting point of the next computation. By this method, a valuable check is secured at the end of each block of five.

* Since on a modern machine division is as easy as multiplication, it may often be convenient to replace the entries opposite $n = 0.5$ in Tables 3 and 3a by the equivalent common fractions, as follows:

(Table 3) $\quad Y_{0.5} = (-Y_{-1} + 9Y_0 + 9Y_1 - Y_2)/16$

(Table 3a) $\quad Y_{0.5} = (5Y_0 + 15Y_1 - 5Y_2 + Y_3)/16$

Many formulas of this type are given in Pearson's Tract.

APPENDIX.

The theory of the Lagrangean interpolation formula is extremely simple. For example, for third degree interpolation in the main part of the table, we form the following function of n:

$$\begin{aligned} F(n) = [-(n)\ (n-1)\ (n-2)/(3!)]\ Y_{-1} \\ + [(n+1)\ (n-1)(n-2)/(1!\,2!)]\ Y_0 \\ + [-(n+1)\ (n)\ (n-2)/(2!\ 1!)]\ Y_1 \\ + [(n+1)\ (n)\ (n-1)/(3!)]\ Y_2 \end{aligned}$$

where Y_{-1}, Y_0, Y_1, Y_2 are given tabular values.

By inspection, we see that this expression is a polynomial of the third degree in n; also, that when $n = -1$, $F(n) = Y_{-1}$; when $n = 0$, $F(n) = Y_0$; when $n = 1$, $F(n) = Y_1$; and when $n = 2$, $F(n) = Y_2$. Hence, when n is a fraction between 0 and 1, $F(n)$ will be a suitable value to take as the interpolated value Y_n. Now the quantities in brackets, [ ], depend only on n, and can be tabulated once for all; these are precisely the quantities given as A_{-1}, A_0, A_1, A_2 in Table 3.

Similarly, for third degree interpolation at the beginning of the table, we form the following function of n:

$$\begin{aligned} F(n) = [-(n-1)\ (n-2)\ (n-3)/(3!)]\ Y_0 \\ + [(n)\ (n-2)\ (n-3)/(1!\,2!)]\ Y_1 \\ + [-(n)\ (n-1)\ (n-3)/(2!\,1!)]\ Y_2 \\ + [(n)\ (n-1)\ (n-2)/(3)]\ Y_3 \end{aligned}$$

and tabulate the coefficients in brackets as a_0, a_1, a_2, a_3 in Table 3a.

If we build analogous expressions for the fourth degree, and subtract from them the expressions just written, we obtain the "corrections" given above.

The formulas underlying Tables 5, 5a, and 5b are obtained in a similar way.

TABLE 3.

FOR 3RD DEGREE INTERPOLATION, IN MAIN PART OF TABLE.

n	A_{-1}	A_0	A_1	A_2
0.00	−0.0000	1.0000	0.0000	−0.0000
.01	−.0032835	0.9945005	.0100495	−.0016665
.02	−.0064680	.9896040	.0201960	−.0033320
.03	−.0095545	.9841135	.0304365	−.0049955
.04	−.0125440	.9784320	.0407680	−.0066560
.05	−.0154375	.9725625	.0511875	−.0083125
.06	−.0182360	.9665080	.0616920	−.0099640
.07	−.0209405	.9602715	.0722785	−.0116095
.08	−.0235520	.9538560	.0829440	−.0132480
.09	−.0260715	.9472645	.0936855	−.0148785
0.10	−0.0285	0.9405	0.1045	−0.0165
.11	−.0308385	.9335655	.1153845	−.0181115
.12	−.0330880	.9264640	.1263360	−.0197120
.13	−.0352495	.9191985	.1373515	−.0213005
.14	−.0373240	.9117720	.1484280	−.0228760
.15	−.0393125	.9041875	.1595625	−.0244375
.16	−.0412160	.8964480	.1707520	−.0259840
.17	−.0430355	.8885565	.1819935	−.0275145
.18	−.0447720	.8805160	.1932840	−.0290280
.19	−.0464265	.8723295	.2046205	−.0305235
0.20	−0.0480	0.8640	0.2160	−0.0320
.21	−.0494935	.8555305	.2274195	−.0334565
.22	−.0509080	.8464240	.2388760	−.0348920
.23	−.0522445	.8381835	.2503665	−.0363055
.24	−.0535040	.8293120	.2618880	−.0376960
.25	−.0546875	.8203125	.2734375	−.0390625
.26	−.0557960	.8111880	.2850120	−.0404040
.27	−.0568305	.8019415	.2966085	−.0417195
.28	−.0577920	.7925760	.3082240	−.0430080
.29	−.0586815	.7830945	.3189555	−.0442685
0.30	−0.0595	0.7735	0.3315	−0.0455
.31	−.0602485	.7637955	.3431545	−.0467015
.32	−.0609280	.7539840	.3548160	−.0478720
.33	−.0615395	.7440685	.3664815	−.0490105
.34	−.0620840	.7340520	.3781480	−.0501160
.35	−.0625625	.7239375	.3898125	−.0511875
.36	−.0629760	.7137280	.4014720	−.0522240
.37	−.0633255	.7034265	.4131235	−.0532245
.38	−.0636120	.6930360	.4247640	−.0541880
.39	−.0638365	.6825595	.4363905	−.0551135
0.40	−0.0640	0.6720	0.4480	−0.0560
.41	−.0641035	.6613605	.4595895	−.0568465
.42	−.0641480	.6506440	.4711560	−.0576520
.43	−.0641345	.6398535	.4826965	−.0584155
.44	−.0640640	.6289920	.4942080	−.0591360
.45	−.0639375	.6180625	.5056875	−.0598125
.46	−.0637560	.6070680	.5171320	−.0604440
.47	−.0635205	.5960115	.5285385	−.0610295
.48	−.0632320	.5848960	.5399040	−.0615680
.49	−.0628915	.5737245	.5512255	−.0620585
0.50	−0.0625	0.5625	0.5625	−0.0625

$$Y_n = A_{-1}Y_{-1} + A_0Y_0 + A_1Y_1 + A_2Y_2$$

TABLE 3 (*continued*).

n	A_{-1}	A_0	A_1	A_2
0.50	−0.0625	0.5625	0.5625	−0.0625
.51	−.0620585	.5512255	.5737245	−.0628915
.52	−.0615680	.5399040	.5848960	−.0632320
.53	−.0610295	.5285385	.5960115	−.0635205
.54	−.0604440	.5171320	.6070680	−.0637560
.55	−.0598125	.5056875	.6180625	−.0639375
.56	−.0591360	.4942080	.6289920	−.0640640
.57	−.0584155	.4826965	.6398535	−.0641345
.58	−.0576520	.4711560	.6506440	−.0641480
.59	−.0568465	.4595895	.6613605	−.0641035
0.60	−0.0560	0.4480	0.6720	−0.0640
.61	−.0551135	.4363905	.6825595	−.0638365
.62	−.0541880	.4247640	.6930360	−.0636120
.63	−.0532245	.4131235	.7034265	−.0633255
.64	−.0522240	.4014720	.7137280	−.0629760
.65	−.0511875	.3898125	.7239375	−.0625625
.66	−.0501160	.3781480	.7340520	−.0620840
.67	−.0490105	.3664815	.7440685	−.0615395
.68	−.0478720	.3548160	.7539840	−.0609280
.69	−.0467015	.3431545	.7637955	−.0602485
0.70	−0.0455	0.3315	0.7735	−0.0595
.71	−.0442685	.3189555	.7830945	−.0586815
.72	−.0430080	.3082240	.7925760	−.0577920
.73	−.0417195	.2966085	.8019415	−.0568305
.74	−.0404040	.2850120	.8111880	−.0557960
.75	−.0390625	.2734375	.8203125	−.0546875
.76	−.0376960	.2618880	.8293120	−.0535040
.77	−.0363055	.2503665	.8381835	−.0522445
.78	−.0348920	.2388760	.8469240	−.0509080
.79	−.0334565	.2274195	.8555305	−.0494935
0.80	−0.0320	0.2160	0.8640	−0.0480
.81	−.0305235	.2046205	.8723295	−.0464265
.82	−.0290280	.1932840	.8805160	−.0447720
.83	−.0275145	.1819935	.8885565	−.0430355
.84	−.0259840	.1707520	.8964480	−.0412160
.85	−.0244375	.1595625	.9041875	−.0393125
.86	−.0228760	.1484280	.9117720	−.0373240
.87	−.0213005	.1373515	.9191985	−.0352495
.88	−.0197120	.1263360	.9264640	−.0330880
.89	−.0181115	.1153845	.9335655	−.0308385
0.90	−0.0165	0.1045	0.9405	−0.0285
.91	−.0148785	.0936855	.9472645	−.0260715
.92	−.0132480	.0829440	.9538560	−.0235520
.93	−.0116095	.0722785	.9602715	−.0209405
.94	−.0099640	.0616920	.9665080	−.0182365
.95	−.0083125	.0511875	.9725625	−.0154375
.96	−.0066560	.0407680	.9784320	−.0125440
.97	−.0049955	.0304365	.9841135	−.0095545
.98	−.0033320	.0201960	.9896040	−.0064680
.99	−.0016665	.0100495	0.9945005	−.0032835
1.00	−0.0000	0.0000	1.0000	−0.0000

Correction required to take into account 4th differences lies between 0 and $+.024\,[Y_{-1} - 4Y_0 + 6Y_1 - 4Y_2 + Y_3.]$

TABLE 3a.

FOR 3RD DEGREE INTERPOLATION, AT BEGINNING OF TABLE.

n	a_0	a_1	a_2	a_3
0.00	1.0000	0.0000	−0.0000	0.0000
.01	.9817665	.0297505	−.0148005	.0032835
.02	.9637320	.0590040	−.0292040	.0064680
.03	.9458955	.0877635	−.0432135	.0095545
.04	.9282560	.1160320	−.0568320	.0125440
.05	.9108125	.1438125	−.0700625	.0154375
.06	.8935640	.1711080	−.0829080	.0182360
.07	.8765095	.1979215	−.0953715	.0209405
.08	.8596480	.2242560	−.1074560	.0235520
.09	.8429785	.2501145	−.1191645	.0260715
0.10	0.8265	0.2755	−0.1305	0.0285
.11	.8102115	.3004155	−.1414655	.0308385
.12	.7941120	.3248640	−.1520640	.0330880
.13	.7782005	.3488485	−.1622985	.0352495
.14	.7624760	.3723720	−.1721720	.0373240
.15	.7469375	.3954375	−.1816875	.0393125
.16	.7315840	.4180480	−.1908480	.0412160
.17	.7164145	.4402065	−.1996565	.0430355
.18	.7014280	.4619160	−.2081160	.0447720
.19	.6866235	.4831795	−.2162295	.0464265
0.20	0.6720	0.5040	−0.2240	0.0480
.21	.6575565	.5243805	−.2314305	.0494935
.22	.6432920	.5443240	−.2385240	.0509080
.23	.6292055	.5638335	−.2452835	.0522445
.24	.6152960	.5829120	−.2517120	.0535040
.25	.6015625	.6015625	−.2578125	.0546875
.26	.5880040	.6197880	−.2635880	.0557960
.27	.5746195	.6375915	−.2690415	.0568305
.28	.5614080	.6549760	−.2741760	.0577920
.29	.5483685	.6719445	−.2789945	.0586815
0.30	0.5355	0.6885	−0.2835	0.0595
.31	.5228015	.7046455	−.2876955	.0602485
.32	.5102720	.7203840	−.2915840	.0609280
.33	.4979105	.7357185	−.2951685	.0615395
.34	.4857160	.7506520	−.2984520	.0620840
.35	.4736875	.7651875	−.3014375	.0625625
.36	.4618240	.7793280	−.3041280	.0629760
.37	.4501245	.7930765	−.3065265	.0633255
.38	.4385880	.8064360	−.3086360	.0636120
.39	.4272135	.8194095	−.3104595	.0638365
0.40	0.4160	0.8320	−0.3120	0.0640
.41	.4049465	.8442105	−.3132605	.0641035
.42	.3940520	.8560440	−.3142440	.0641480
.43	.3833155	.8675035	−.3149535	.0641345
.44	.3727360	.8785920	−.3153920	.0640640
.45	.3623125	.8893125	−.3155625	.0639375
.46	.3520440	.8996680	−.3154680	.0637560
.47	.3419295	.9096615	−.3151115	.0635205
.48	.3319680	.9192960	−.3144960	.0632320
.49	.3221585	.9285745	−.3136245	.0628915
0.50	0.3125	0.9375	−0.3125	0.0625

$$Y_n = a_0 Y_0 + a_1 Y_1 + a_2 Y_2 + a_3 Y_3$$

TABLE 3a (*continued*).

n	a_0	a_1	a_2	a_3
0.50	0.3125	0.9375	−0.3125	0.0625
.51	.3029915	.9460755	−.3111255	.0620585
.52	.2936320	.9543040	−.3095040	.0615680
.53	.2844205	.9621885	−.3076385	.0610295
.54	.2753560	.9697320	−.3055320	.0604440
.55	.2664375	.9769375	−.3031875	.0598125
.56	.2576640	.9838080	−.3006080	.0591360
.57	.2490345	.9903465	−.2977965	.0584155
.58	.2405480	0.9965560	−.2947560	.0576520
.59	.2322035	1.0024395	−.2914895	.0568465
0.60	0.2240	1.0080	−0.2880	0.0560
.61	.2159365	1.0132405	−.2842905	.0551135
.62	.2080120	1.0181640	−.2803640	.0541880
.63	.2002255	1.0227735	−.2762235	.0532245
.64	.1925760	1.0270720	−.2718720	.0522240
.65	.1850625	1.0310625	−.2673125	.0511875
.66	.1776840	1.0347480	−.2625480	.0501160
.67	.1704395	1.0381315	−.2575815	.0490105
.68	.1633280	1.0412160	−.2524160	.0478720
.69	.1563485	1.0440045	−.2470545	.0467015
0.70	0.1495	1.0465	−0.2415	0.0455
.71	.1427815	1.0487055	−.2357555	.0442685
.72	.1361920	1.0506240	−.2298240	.0430080
.73	.1297305	1.0522585	−.2237085	.0417195
.74	.1233960	1.0536120	−.2174120	.0404040
.75	.1171875	1.0546875	−.2109375	.0390625
.76	.1111040	1.0554880	−.2042880	.0376960
.77	.1051445	1.0560165	−.1974665	.0363055
.78	.0993080	1.0562760	−.1904760	.0348920
.79	.0935935	1.0562695	−.1833195	.0334565
0.80	0.0880	1.0560	−0.1760	0.0320
.81	.0825265	1.0554705	−.1685205	.0305235
.82	.0771720	1.0546840	−.1608840	.0290280
.83	.0719355	1.0536435	−.1530935	.0275145
.84	.0668160	1.0523520	−.1451520	.0259840
.85	.0618125	1.0508125	−.1370625	.0244375
.86	.0569240	1.0490280	−.1288280	.0228760
.87	.0521495	1.0470015	−.1204515	.0213005
.88	.0474880	1.0447360	−.1119360	.0197120
.89	.0429385	1.0422345	−.1032845	.0181115
0.90	0.0385	1.0395	−0.0945	0.0165
.91	.0341715	1.0365355	−.0855855	.0148785
.92	.0299520	1.0333440	−.0765440	.0132480
.93	.0258405	1.0299285	−.0673785	.0116095
.94	.0218360	1.0262920	−.0580920	.0099640
.95	.0179375	1.0224375	−.0486875	.0083125
.96	.0141440	1.0183680	−.0391680	.0066560
.97	.0104545	1.0140865	−.0295365	.0049955
.98	.0068680	1.0095960	−.0197960	.0033320
.99	.0033835	1.0048995	−.0099495	.0016665
1.00	0.0000	1.0000	−0.0000	0.0000

Correction required to take into account 4th differences lies between 0 and $-.042\,[Y_0 - 4Y_1 + 6Y_2 - 4Y_3 + Y_4.]$

TABLE 5.

FOR 5TH DEGREE INTERPOLATION, IN MAIN PART OF TABLE

n	B_{-2}	B_{-1}	B_0	B_1	B_2	B_3
0.00	0.000000	−0.000000	1.000000	0.000000	−0.000000	0.000000
.01	.00049579	−.00493338	0.99654209	.01006608	−.00250387	.00033329
.02	.00098301	−.00973369	.99283671	.02026197	−.00501433	.00066633
.03	.00146141	−.01440126	.98888645+	.03058412	−.00752959	.00099888
.04	.00193077	−.01893642	.98469396	.04102892	−.01004790	.00133067
.05	.00239088	−.02333957	.98026195+	.05159273	−.01256746	.00166146
.06	.00284153	−.02761113	.97559318	.06227190	−.01508649	.00199101
.07	.00328153	−.03175156	.97069045−	.07306272	−.01760319	.00231906
.08	.00371368	−.03576136	.96555663	.08396145−	−.02011576	.00264536
.09	.00413481	−.03964106	.96019466	.09496431	−.02262239	.00296967
0.10	0.00454575	−0.04339125	0.95460750	0.10606750	−0.02512125	0.00329175
.11	.00494634	−.04701252	.94879818	.11726719	−.02761053	.00361134
.12	.00533643	−.05050552	.94276977	.12855951	−.03008840	.00392821
.13	.00571588	−.05387093	.93652539	.13994058	−.03255302	.00424210
.14	.00608456	−.05710945+	.93006822	.15140646	−.03500257	.00455278
.15	.00644234	−.06022184	.92340148	.16295320	−.03743520	.00486001
.16	.00678910	−.06320886	.91652844	.17457684	−.03984906	.00516354
.17	.00712474	−.06607133	.90945239	.18627338	−.04224232	.00546314
.18	.00744917	−.06881009	.90217669	.19803879	−.04461313	.00575857
.19	.00776228	−.07142601	.89470475+	.20986902	−.04695964	.00604961
0.20	0.008064	−0.073920	0.887040	0.221760	−0.049280	0.006336
.21	.00835426	−.07629299	.87918592	.23370765−	−.05157296	.00661753
.22	.00863298	−.07854595+	.87114603	.24570785+	−.05383487	.00689396
.23	.00890011	−.08067988	.86292388	.25775648	−.05606568	.00716507
.24	.00915560	−.08269578	.85452308	.26984940	−.05826294	.00743064
.25	.00939941	−.08459473	.84594727	.28198242	−.06042480	.00769043
.26	.00963151	−.08637779	.83720010	.29415138	−.06254943	.00794424
.27	.00985185+	−.08804607	.82828528	.30635209	−.06463498	.00819183
.28	.01006143	−.08960072	.81920655+	.31858033	−.06667960	.00843301
.29	.01025723	−.09104288	.80996769	.33083187	−.06868147	.00866755+
0.30	0.01044225	−0.09237375	0.80057250	0.34310250	−0.07063875	0.00889525
.31	.01061548	−.09359454	.79102481	.35538796	−.07254961	.00911590
.32	.01077694	−.09470648	.78132849	.36768399	−.07441224	.00932930
.33	.01092665−	−.09571085−	.77148742	.37998634	−.07622481	.00953524
.34	.01106461	−.09660891	.76150554	.39229074	−.07798551	.00973353
.35	.01119087	−.09740199	.75138680	.40459289	−.07969254	.00992398
.36	.01130545+	−.09809142	.74113516	.41688852	−.08134410	.01010639
.37	.01140841	−.09867854	.73075462	.42917335−	−.08293841	.01028058
.38	.01149978	−.09916475−	.72024921	.44144307	−.08447367	.01044636
.39	.01157962	−.09955143	.70962298	.45369338	−.08594813	.01060356
0.40	0.011648	−0.099840	0.698880	0.465920	−0.087360	0.010752
.41	.01170498	−.10003191	.68802435+	.47811862	−.08870754	.01089151
.42	.01175063	−.10012861	.67706015−	.49028493	−.08998901	.01102191
.43	.01178504	−.10013159	.66599152	.50241465+	−.09120266	.01114305−
.44	.01180828	−.10004234	.65482260	.51450348	−.09234678	.01125476
.45	.01182045−	−.09986238	.64355758	.52654711	−.09341965−	.01135690
.46	.01182164	−.09959325−	.63220062	.53854126	−.09441957	.01144930
.47	.01181195+	−.09923649	.62075591	.55048166	−.09534486	.01153183
.48	.01179150+	−.09879368	.60922767	.56236401	−.09619384	.01160434
.49	.01176040	−.09826640	.59762013	.57418404	−.09696485+	.01166669
0.50	0.01171875	−0.09765625	0.58593750	0.58593750	−0.09765625	0.01171875

$$Y_n = B_{-2}Y_{-2} + B_{-1}Y_{-1} + B_0Y_0 + B_1Y_1 + B_2Y_2 + B_3Y_3$$

TABLE 5 (*continued*).

n	B_{-2}	B_{-1}	B_0	B_1	B_2	B_3
.50	0.01171875	−0.09765625	0.58593750	0.58593750	−0.09765625	0.01171875
.51	.01166669	−.09696485+	.57418404	.59762013	−.09826640	.01176040
.52	.01160434	−.09619384	.56236401	.60922767	−.09879368	.01179150+
.53	.01153183	−.09534486	.55048166	.62075591	−.09923649	.01181195+
.54	.01144930	−.09441957	.53854126	.63220062	−.09959325−	.01182164
.55	.01135690	−.09341965−	.52654711	.64355758	−.09986238	.01182045−
.56	.01125476	−.09234678	.51450348	.65482260	−.10004234	.01180828
.57	.01114305−	−.09120266	.50241465+	.66599152	−.10013159	.01178504
.58	.01102191	−.08998901	.49028493	.67706015−	−.10012861	.01175063
.59	.01089151	−.08870754	.47811862	.68802435+	−.10003191	.01170498
.60	0.010752	−0.087360	0.465920	0.698880	−0.099840	0.011648
.61	.01060356	−.08594813	.45369338	.70962298	−.09955143	.01157962
.62	.01044636	−.08447367	.44144307	.72024921	−.09916475−	.01149978
.63	.01028058	−.08293841	.42917335−	.73075462	−.09867854	.01140841
.64	.01010639	−.08134410	.41688852	.74113516	−.09809142	.01130545+
.65	.00992398	−.07969254	.40459289	.75138680	−.09740199	.01119087
.66	.00973353	−.07798551	.39229074	.76150554	−.09660891	.01106461
.67	.00953524	−.07622481	.37998634	.77148742	−.09571085−	.01092665−
.68	.00932930	−.07441224	.36768399	.78132849	−.09470648	.01077694
.69	.00911590	−.07254961	.35538796	.79102481	−.09359454	.01061548
.70	0.00889525	−0.07063875	0.34310250	0.80057250	−0.09237375	0.01044225
.71	.00866755+	−.06868147	.33083187	.80996769	−.09104288	.01025723
.72	.00843301	−.06667960	.31858033	.81920655+	−.08960072	.01006143
.73	.00819183	−.06463498	.30635209	.82828528	−.08804607	.00985185+
.74	.00794424	−.06254943	.29415138	.83720010	−.08637779	.00963151
.75	.00769043	−.06042480	.28198242	.84594727	−.08459473	.00939941
.76	.00743064	−.05826294	.26984940	.85452308	−.08269578	.00915560
.77	.00716507	−.05606568	.25775648	.86292388	−.08067988	.00890011
.78	.00689396	−.05383487	.24570785+	.87114603	−.07854595+	.00863298
.79	.00661753	−.05157296	.23370765−	.87918592	−.07629299	.00835426
.80	0.006336	−0.049280	0.221760	0.887040	−0.073920	0.008064
.81	.00604961	−.04695964	.20986902	.89470475+	−.07142601	.00776228
.82	.00575857	−.04461313	.19803879	.90217669	−.06881009	.00744917
.83	.00546314	−.04224232	.18627338	.90945239	−.06607133	.00712474
.84	.00516354	−.03984906	.17457684	.91652844	−.06320886	.00678910
.85	.00486001	−.03743520	.16295320	.92340148	−.06022184	.00644234
.86	.00455278	−.03500257	.15140646	.93006822	−.05710945+	.00608456
.87	.00424210	−.03255302	.13994058	.93652539	−.05387093	.00571588
.88	.00392821	−.03008840	.12855951	.94276977	−.05050552	.00533643
.89	.00361134	−.02761053	.11726719	.94879818	−.04701252	.00494634
.90	0.00329175	−0.02512125	0.10606750	0.95460750	−0.04339125	0.00454575
.91	.00296967	−.02262239	.09496431	.96019466	−.03964106	.00413481
.92	.00264536	−.02011576	.08396145−	.96555663	−.03576136	.00371368
.93	.00231906	−.01760319	.07306272	.97069045−	−.03175156	.00328153
.94	.00199101	−.01508649	.06227190	.97559318	−.02761113	.00284153
.95	.00166146	−.01256746	.05159273	.98026195+	−.02333957	.00239088
.96	.00133067	−.01004790	.04102892	.98469396	−.01893642	.00193077
.97	.00099888	−.00752959	.03058412	.98888645+	−.01440126	.00146141
.98	.00066633	−.00501433	.02026197	.99283671	−.00973369	.00098301
.99	.00033329	−.00250387	.01006608	0.99654209	−.00493338	.00049579
1.00	0.000000	−0.000000	0.000000	1.000000	−0.000000	0.000000

Correction required to take into account 6th differences lies between zero and $-0.005\,[Y_{-2} - 6Y_{-1} + 15Y_0 - 20Y_1 + 15Y_2 - 6Y_3 + Y_4]$

TABLE 5a.

FOR 5TH DEGREE INTERPOLATION, AT BEGINNING OF TABLE.

n	b_0	b_1	b_2	b_3	b_4	b_5
0.00	1.000000	0.000000	−0.000000	0.000000	−0.000000	0.000000
.01	0.97735398	.04936129	−.04911324	.03268741	−.01224754	.00195862
.02	.95507820	.09745691	−.09647249	.06409917	−.02399693	.00383563
.03	.93316907	.14430440	−.14210687	.09425944	−.03525825+	.00563281
.04	.91162225+	.18992118	−.18604524	.12319212	−.04604150−	.00735179
.05	.89043862	.23432449	−.22831617	.15092086	−.05635652	.00899427
.06	.86959907	.27753147	−.26894802	.17746910	−.06621309	.01056193
.07	.84911454	.31955911	−.30796888	.20286005−	−.07562086	.01205639
.08	.82897598	.36042424	−.34540657	.22711665−	−.08458936	.01347928
.09	.80917938	.40014360	−.38128866	.25026163	−.09312805−	.01483221
0.10	0.78972075	0.43873375	−0.41564250	0.27231750	−0.10124625	0.01611675
.11	.77059653	.47621114	−.44849515−	.29330652	−.10895319	.01733448
.12	.75180222	.51259208	−.47987343	.31325071	−.11625800	.01848693
.13	.73333394	.54789273	−.50980393	.33217190	−.12316968	.01957563
.14	.71518784	.58212915−	−.53831298	.35009166	−.12969717	.02060210
.15	.69736007	.61531723	−.56542664	.36703133	−.13584926	.02156782
.16	.67984684	.64747274	−.59117076	.38301204	−.14163466	.02247426
.17	.66264436	.67861134	−.61557094	.39805470	−.14706199	.02332287
.18	.64594887	.70874851	−.63865251	.41217998	−.15213973	.02411509
.19	.62915665+	.73789965+	−.66044057	.42540834	−.15687630	.02485324
0.20	0.612864	0.766080	−0.680960	0.437760	−0.161280	0.025536
.21	.59686765−	.79330467	−.70023541	.44925497	−.16535902	.02616746
.22	.58116337	.81958865−	−.71829117	.45991305+	−.16912147	.02674808
.23	.56574757	.84494679	−.73515144	.46975381	−.17257534	.02727920
.24	.55061668	.86939382	−.75084012	.47879660	−.17572854	.02776216
.25	.53576715−	.89294434	−.76538086	.48706055−	−.17858887	.02819824
.26	.52119544	.91561281	−.77879710	.49456458	−.18116403	.02858875
.27	.50689806	.93741359	−.79111205−	.50132741	−.18346164	.02893496
.28	.49287154	.95836088	−.80234865−	.50736753	−.18544920	.02923813
.29	.47911241	.97846878	−.81252963	.51270320	−.18725413	.02949948
0.30	0.46561725	0.99775125	−0.82167750	0.51735250	−0.18876375	0.02972025
.31	.45238307	1.01622212	−.82981452	.52133328	−.19002528	.02990163
.32	.43940594	1.03389512	−.83696271	.52466319	−.19104584	.03004482
.33	.42668251	1.05078382	−.84314390	.52735967	−.19183247	.03015097
.34	.41420949	1.06690169	−.84837966	.52943994	−.19239211	.03022125
.35	.40198360	1.08226207	−.85269133	.53092102	−.19273160	.03025679
.36	.38000159	1.09687818	−.85610004	.53181972	−.19285770	.03025871
.37	.36826022	1.11076312	−.85862671	.53215267	−.19277707	.03022811
.38	.35675628	1.12392985+	−.86029199	.53193627	−.19249627	.03016608
.39	.34544866	1.13639124	−.86111634	.53118671	−.19202179	.03007369
0.40	0.344448	1.148160	−0.861120	0.529920	−0.191360	0.029952
.41	.33363772	1.15924875+	−.86032297	.52815194	−.19051720	.02980204
.42	.32305222	1.16966999	−.85874505+	.52589813	−.18949961	.02962483
.43	.31268830	1.17943607	−.85640581	.52317398	−.18831332	.02942138
.44	.30254296	1.18855926	−.85332460	.51999468	−.18696438	.02919268
.45	.29261318	1.19705168	−.84952055−	.51637523	−.18545871	.02893971
.46	.28289597	1.20492535+	−.84501258	.51233046	−.18380217	.02866342
.47	.27338837	1.21219217	−.83981941	.50787498	−.18200052	.02836476
.48	.26408746	1.21886392	−.83395953	.50302321	−.18005944	.02804466
.49	.25499017	1.22495227	−.82745120	.49778937	−.17798452	.02770402
0.50	0.24609375	1.23046875	−0.82031250	0.49218750	−0.17578125	0.02734375

$$Y_n = b_0 Y_0 + b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y_4 + b_5 Y_5$$

TABLE 5a (*Continued*).

n	b_0	b_1	b_2	b_3	b_4	b_5
.50	0.24609375	1.23046875	−0.82031250	0.49218750	−0.17578125	0.02734375
.51	.23739568	1.23542481	− .81256128	.48623145+	− .17345506	.02696523
.52	.22889253	1.23983176	− .80421519	.47993487	− .17101128	.02656782
.53	.22058149	1.24370080	− .79529167	.47331124	− .16845515	.02615389
.54	.21245978	1.24704303	− .78580794	.46637382	− .16579185−	.02572376
.55	.20452463	1.24986941	− .77578102	.45913570	− .16302645−	.02527826
.56	.19677330	1.25219082	− .76522772	.45160980	− .16016394	.02481820
.57	.18920306	1.25401800	− .75416467	.44380884	− .15720925	.02434437
.58	.18181121	1.25536159	− .74260827	.43574535−	− .15416721	.02885755+
.59	.17459508	1.25623212	− .73057471	.42743168	− .15104257	.02385851
.60	0.167552	1.256640	−0.718080	0.418880	−0.147840	0.022848
.61	.16067975+	1.25659564	− .70513994	.41010231	− .14456409	.02232675+
.62	.15397515+	1.25610893	− .69177013	.40111041	− .14121935−	.02179550−
.63	.14743564	1.25519025+	− .67798598	.39191594	− .13781021	.02125494
.64	.14105867	1.25384950−	− .66380268	.38253036	− .13434102	.02070577
.65	.13484171	1.25209652	− .64923523	.37296492	− .13081605+	.02014868
.66	.12878227	1.24994109	− .63429846	.36323074	− .12723951	.01958433
.67	.12287785+	1.24739286	− .61900698	.35333875−	− .12361551	.01901338
.68	.11712600	1.24446136	− .60337521	.34329969	− .11994808	.01843646
.69	.11152427	1.24115605−	− .58741737	.33312413	− .11624120	.01785422
.70	0.10607025	1.23748625	−0.57114750	0.32282250	−0.11249875	0.01726725
.71	.10076194	1.23346119	− .55457945+	.31240502	− .10872454	.01667617
.72	.09559640	1.22909000	− .53772687	.30188175+	− .10492232	.01608155+
.73	.09057129	1.22438168	− .52060324	.29126260	− .10109574	.01548398
.74	.08568431	1.21934517	− .50322182	.28055730	− .09724839	.01488403
.75	.08093316	1.21398926	− .48559570	.26077539	− .09338399	.01428223
.76	.07631557	1.20832266	− .46773780	.25892628	− .08950538	.01367912
.77	.07182928	1.20235399	− .44966084	.24801921	− .08561654	.01307524
.78	.06747207	1.19609173	− .43137735−	.23706323	− .08172055+	.01247110
.79	.06324170	1.18954430	− .41289968	.22606724	− .07782066	.01186719
.80	0.059136	1.182720	−0.394240	0.215040	−0.073920	0.011264
.81	.05515319	1.17562702	− .37541031	.20399008	− .07002167	.01066201
.82	.05129057	1.16827347	− .35642241	.19292589	− .06612869	.01006169
.83	.04754600	1.16066734	− .33728794	.18185571	− .06224399	.00946347
.84	.04391742	1.15281654	− .31801836	.17078764	− .05837046	.00886782
.85	.04040274	1.14472887	− .29862492	.15972961	− .05451090	.00827515−
.86	.03699992	1.13641203	− .27911874	.14868942	− .05066805+	.00768588
.87	.03370691	1.12787364	− .25951075−	.13767472	− .04684459	.00710042
.88	.03052171	1.11912120	− .23981169	.12669297	− .04304312	.00651915+
.89	.02744232	1.11016213	− .22003213	.11575150+	− .03926618	.00594247
.90	0.02446675	1.10100375	−0.20018250	0.10485750	−0.03551625	0.00537075
.91	.02159346	1.09165328	− .18027302	.09402005+	− .03179573	.00480434
.92	.01881995+	1.08211784	− .16031375+	.08324319	− .02810696	.00424360
.93	.01614432	1.07240447	− .14031460	.07253379	− .02445222	.00368886
.94	.01356469	1.06252011	− .12028530	.06189354	− .02083373	.00314045+
.95	.01107920	1.05247160	− .10023539	.05134398	− .01725363	.00259870
.96	.00868600	1.04226570	− .08017428	.04087652	− .01371402	.00206389
.97	.00638329	1.03190907	− .06011121	.03050239	− .01021692	.00153634
.98	.00416924	1.02140827	− .04005523	.00022767	− .00676429	.00101633
.99	.00204207	1.01076979	− .02001524	.01005828	− .00335804	.00050412
1.00	0.000000	1.000000	−0.000000	0.000000	−0.000000	0.000000

Correction required to take into account 6th differences lies between zero and

$$-0.024\,[Y_0 - 6Y_1 + 15Y_2 - 20Y_3 + 15Y_4 - 6Y_5 + Y_6]$$

TABLE 5b.

FOR 5TH DEGREE INTERPOLATION, IN SECOND INTERVAL OF TABLE.

n	c_{-1}	c_0	c_1	c_2	c_3	c_4
0.00	−0.000000	1.000000	0.000000	−0.000000	0.000000	−0.000000
.01	−.00195862	0.98910520	.01998192	−.00994076	.00330966	−.00049579
.02	−.00383565+	.97809161	.03992211	−.01975943	.00656693	−.00098301
.03	−.00563281	.96696532	.05981229	−.03945052	.00977030	−.00146141
.04	−.00735179	.95573228	.07964436	−.03900948	.01291830	−.00193077
.05	−.00899427	.94439871	.09941039	−.04843070	.01600949	−.00239088
.06	−.01056193	.93297017	.11910258	−.05770950−	.01904249	−.00284153
.07	−.01205639	.92145252	.13871328	−.06684112	.02201595+	−.00328253
.08	−.01347928	.90985144	.15823503	−.07582095+	.02492856	−.00371368
.09	−.01483221	.89817252	.17766050−	−.08464453	.02777907	−.00463481
0.10	−0.01611675	0.88642125	0.19698250	−0.09330750	0.03056625	−0.00454575
.11	−.01733448	.87460306	.21619401	−.10180565−	.03329100	−.00494634
.12	−.01848693	.86272328	.23528817	−.11013489	.03594936	−.00533643
.13	−.01957563	.85078715+	.25425823	−.11829126	.03854037	−.00571588
.14	−.02060210	.83879985−	.27300762	−.12629094	.04106313	−.00 08456
.15	−.02156782	.82676645−	.29179992	−.13407023	.04351676	−.00647234
.16	−.02247426	.81469194	.31035884	−.14168556	.04590042	−.00678910
.17	−.02332287	.80258125+	.32876822	−.14911346	.04821332	−.00712474
.18	−.02411509	.79043921	.34702209	−.15635061	.05045469	−.00744917
.19	−.02485324	.77827057	.36511459	−.16339382	.05262381	−.00776228
0.20	−0.025536	0.766080	0.383040	−0.170240	0.054720	−0.008064
.21	−.02616746	.75387209	.40079276	−.17688619	.05674467	−.00835426
.22	−.02674808	.74165135−	.41836743	−.18332955−	.05869439	−.00863298
.23	−.02727920	.72942221	.43575872	−.18956735+	.06056871	−.00890011
.24	−.02776216	.71718902	.45296148	−.19559700	.06236726	−.00915560
.25	−.02819824	.70495605+	.46997070	−.20141602	.06408965−	−.00939941
.26	−.02858875+	.69272751	.48678150−	−.20702202	.06573555+	−.00963151
.27	−.02893496	.68050751	.50338912	−.21241275−	.06730467	−.00985185
.28	−.02923813	.66830008	.51978895+	−.21758607	.06879672	−.01006143
.29	−.02949948	.65610921	.53597653	−.22253996	.07021148	−.01025723
0.30	−0.02972025	0.64393875	0.55194750	−0.22727250	0.07154875	−0.01044225
.31	−.02990163	.63179254	.56769765	−.23178188	.07281042	−.01061548
.32	−.03004482	.61967432	.58322289	−.23606641	.07399352	−.01077694
.33	−.03015097	.60758774	.59851926	−.24012449	.07509809	−.01092665
.34	−.03022125−	.59553639	.61358294	−.24395466	.07612419	−.01106461
.35	−.03025679	.58352379	.62841023	−.24755555−	.07707191	−.01119087
.36	−.03025871	.57155338	.64299756	−.25092588	.07794138	−.01130545
.37	−.03022811	.55962854	.65734146	−.25406449	.07873274	−.01140841
.38	−.03016608	.54775255+	.67143861	−.25697033	.07944615+	−.01149978
.39	−.03007369	.53592866	.68528582	−.25964245+	.08008183	−.01157962
0.40	−0.029952	0.524160	0.698880	−0.262080	0.080640	−0.011648
.41	−.02980204	.51244967	.71221819	−.26428222	.08112299	−.01170498
.42	−.02962483	.50080069	.72529755−	−.26624847	.08152825−	−.01175063
.43	−.02942138	.48921599	.73811535+	−.26797819	.08185623	−.01178504
.44	−.02919268	.47769846	.75066900	−.26947092	.08210742	−.01180828
.45	−.02893971	.46625090	.76295602	−.27072633	.08228230	−.01182045
.46	−.02866342	.45487605+	.77497402	−.29174414	.08238141	−.01182164
.47	−.02836476	.44357659	.78672075−	−.27252418	.08240528	−.01181195
.48	−.02804466	.43235512	.79819407	−.27306639	.08235448	−.01179150
.49	−.02770402	.42121418	.80939196	−.27337080	.08222960	−.01176040
0.50	−0.02734375	0.41015625	0.82031250	−0.27343750	0.08203125	−0.01171875

$$Y_n = c_{-1}Y_{-1} + c_0Y_0 + c_1Y_1 + c_2Y_2 + c_3Y_3 + c_4Y_4$$

Table 5b (*continued*).

n	c_{-1}	c_0	c_1	c_2	c_3	c_4
0.50	−0.02734375	0.41015625	0.82031250	−0.27343750	0.08203125	−0.01171875
.51	−.02696523	.39918373	.83095388	−.27326671	.08176213	−.01166669
.52	−.02656982	.38829896	.84131441	−.27285873	.08142008	−.01160434
.53	−.02615389	.37750422	.85139249	−.27221393	.08100590	−.01153183
.54	−.02572376	.36680173	.86118666	−.27133278	.08052045−	−.01144930
.55	−.02527826	.35619363	.87069555−	−.27021586	.07996457	−.01135690
.56	−.02481820	.34568202	.87991788	−.26886380	.07933914	−.01125476
.57	−.02434437	.33526892	.88885249	−.26727732	.07864505+	−.01114305−
.58	−.02385755+	.32495629	.89749833	−.26545725+	.07788321	−.01102191
.59	−.02335851	.31474604	.90585445+	−.26340449	.07705455−	−.01089151
0.60	−0.022848	0.304640	0.913920	−0.261120	0.076160	−0.010752
.61	−.02232675+	.29463996	.92169422	−.25860485+	.07520260	−.01060356
.62	−.02179550−	.28474763	.92917647	−.25586019	.07418051	−.01044636
.63	−.02125494	.27496467	.93636619	−.25288722	.07309485	−.01028059
.64	−.02070577	.26529270	.94326292	−.24968724	.07194678	−.01010639
.65	−.02014868	.25593324	.94986633	−.24626164	.07073746	−.00992398
.66	−.01958433	.24628779	.95617614	−.24261186	.06946807	−.00973353
.67	−.01901338	.23695970	.96219218	−.23873941	.06813980	−.00953524
.68	−.01843646	.22974456	.96991439	−.23464591	.0675384	−.00932930
.69	−.01785422	.21864947	.97334280	−.23033303	.06531142	−.00911590
0.70	−0.01726725	0.20967375	0.97847750	−0.22580250	0.06381375	−0.00889525
.71	−.01667617	.20081861	.98331871	−.22105614	.06226415+	−.00866755+
.72	−.01608155+	.19208520	.98786673	−.21609585−	.06066104	−.00843301
.73	−.01548398	.18347460	.99212193	−.21092356	.05900582	−.00819183
.74	−.01488403	.17498787	.99608478	−.20554130	.05729991	−.00794424
.75	−.01428223	.16662598	0.99975586	−.19995117	.05554473	−.00769043
.76	−.01367912	.15838986	1.00313580	−.19415532	.05374170	−.00743064
.77	−.01307524	.15028041	1.00622532	−.18815595+	.05189227	−.00716507
.78	−.01247110	.14229843	1.00902525+	−.18195537	.04999783	−.00689396
.79	−.01186719	.13444472	1.01153649	−.17555592	.04805996	−.00661753
0.80	−0.011264	0.126720	1.013760	−0.168960	0.046080	−0.006336
.81	−.01066201	.11912494	1.01569685+	−.16217009	.04406152	−.00604961
.82	−.01006169	.11166019	1.01734819	−.15518871	.04200317	−.00575857
.83	−.00946347	.10432626	1.01871522	−.14801845+	.03990656	−.00546314
.84	−.00886782	.09712374	1.01979924	−.14066196	.03777334	−.00516354
.85	−.00827515−	.09005809	1.02060164	−.13312195+	.03560512	−.00486001
.86	−.00768588	.08311473	1.02112386	−.12540118	.03340353	−.00455278
.87	−.00710042	.07630906	1.02136741	−.11750245−	.03117022	−.00424210
.88	−.00651915+	.06963640	1.02133391	−.10942863	.02890680	−.00392821
.89	−.00594247	.06309705+	1.02102503	−.10118266	.02661493	−.00361134
0.90	−0.00537075	0.05669125	1.02044250	−0.09276750	0.02429625	−0.00329175
.91	−.00480434	.05041919	1.01958814	−.08418618	.02195447	−.00296967
.92	−.00424360	.04428104	1.01846385−	−.07544177	.01958840	−.00264536
.93	−.00368886	.03829689	1.01707156	−.06653739	.01719983	−.00231906
.94	−.00314045+	.03240681	1.01541330	−.05747622	.01479057	−.00199101
.95	−.00259870	.02667082	1.01349117	−.04826148	.01236238	−.00166146
.96	−.00206389	.02106890	1.01130732	−.03889644	.00991706	−.00133067
.97	−.00153634	.01560099	1.00886395+	−.02938439	.00745639	−.00099888
.98	−.00101633	.01026697	1.00616337	−.01972869	.00498213	−.00066633
.99	−.00050412	.00506671	1.00320792	−.00993275+	.00249608	−.00033329
1.00	−0.000000	0.000000	1.000000	−0.000000	0.000000	−0.000000

Correction required to take into account 6th differences lies between zero and
$+ 0.007\ [Y_{-1} - 6Y_0 + 15Y_1 - 20Y_2 + 15Y_3 - 6Y_4 + Y_5]$

TABLE 30a.

(CUMULATIVE) 3RD DEGREE. FOR USE AT BEGINNING OF TABLE.

m	d_0	d_1	d_2	d_3	
0	−0.000	+0.000	−0.000	+0.000	
1/5	−0.328	+0.504	−0.224	+0.048	$Y_m = Y_{m-1/5} + d_0 Y_0 + d_1 Y_1 + d_2 Y_2 + d_3 Y_3$
2/5	−0.256	+0.328	−0.088	+0.016	$(m = 1/5, 2/5, 3/5, 4/5, 1)$
3/5	−0.192	+0.176	+0.024	−0.008	
4/5	−0.136	+0.048	+0.112	−0.024	
1	−0.088	−0.056	+0.176	−0.032	

TABLE 30.

(CUMULATIVE) 3RD DEGREE. FOR USE IN MAIN PART OF TABLE.

m	D_{-1}	D_0	D_1	D_2	
0	−0.000	−0.000	+0.000	−0.000	
1/5	−0.048	−0.136	+0.216	−0.032	$Y_m \quad Y_{m-1/5} + D_{-1} Y_{-1} + D_0 Y_0 + D_1 Y_1 + D_2 Y_2$
2/5	−0.016	−0.192	+0.232	−0.024	$(m = 1/5, 2/5, 3/5, 4/5, 1)$
3/5	+0.008	−0.224	+0.224	−0.008	
4/5	+0.024	−0.232	+0.192	+0.016	
1	+0.032	−0.216	+0.136	+0.048	

TABLE 50a.

(CUMULATIVE) 5TH DEGREE. FOR USE AT BEGINNING OF TABLE.

m	g_0	g_1	g_2	g_3	g_4	g_5
0	−0.000000	+0.000000	−0.000000	+0.000000	−0.000000	+0.000000
1/5	−0.387136	+0.766080	−0.680960	+0.437760	−0.161280	+0.025536
2/5	−0.268416	+0.382080	−0.180160	+0.092160	−0.030080	+0.004416
3/5	−0.176896	+0.108480	+0.143040	−0.111040	+0.043520	−0.007104
4/5	−0.108416	−0.073920	+0.323840	−0.203840	+0.073920	−0.011584
1	−0.059136	−0.182720	+0.394240	−0.215040	+0.073920	−0.011264

$$Y_m = Y_{m-1/5} + g_0 Y_0 + g_1 Y_1 + g_2 Y_2 + g_3 Y_3 + g_4 Y_4 + g_5 Y_5 \qquad (m = 1/5, 2/5, 3/5, 4/5, 1)$$

TABLE 50b.

(CUMULATIVE) 5TH DEGREE. FOR USE IN SECOND INTERVAL OF TABLE.

m	h_{-1}	h_0	h_1	h_2	h_3	h_4
0	−0.000000	−0.000000	+0.000000	−0.000000	+0.000000	−0.000000
$\frac{1}{5}$	−0.025536	−0.233920	+0.383040	−0.170240	+0.054720	−0.008064
$\frac{2}{5}$	−0.004416	−0.241920	+0.315840	−0.091840	+0.025920	−0.003584
$\frac{3}{5}$	+0.007104	−0.219520	+0.215040	+0.000960	−0.004480	+0.000896
$\frac{4}{5}$	+0.011584	−0.177920	+0.099840	+0.092160	−0.030080	+0.004416
1	+0.011264	−0.126720	−0.013760	+0.168960	−0.046080	+0.006336

$Y_m = Y_{m-1/5} + h_{-1}Y_{-1} + h_0Y_0 + h_1Y_1 + h_2Y_2 + h_3Y_3 + h_4Y_4 \quad (m = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1)$

TABLE 50.

(CUMULATIVE) 5TH DEGREE. FOR USE IN MAIN PART OF TABLE.

m	G_{-2}	G_{-1}	G_0	G_1	G_2	G_3
0	+0.000000	−0.000000	−0.000000	+0.000000	−0.000000	+0.000000
$\frac{1}{5}$	+0.008064	−0.073920	−0.112960	+0.221760	−0.049280	+0.006336
$\frac{2}{5}$	+0.003584	−0.025920	−0.188160	+0.244160	−0.038080	+0.004416
$\frac{3}{5}$	−0.000896	+0.012480	−0.232960	+0.232960	−0.012480	+0.000896
$\frac{4}{5}$	−0.004416	+0.038080	−0.244160	+0.188160	+0.025920	−0.003584
1	−0.006336	+0.049280	−0.221760	+0.112960	+0.073920	−0.008064

$Y_m = Y_{m-1/5} + G_{-2}Y_{-2} + G_{-1}Y_{-1} + G_0Y_0 + G_1Y_1 + G_2Y_2 + G_3Y_3 \quad (m = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1)$